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The Precise Value of Commutativity Degree in Some Finite Groups

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ABSTRACT

The commutativity degree of a finite group G , denoted by $P(G)$, is the probability that a selected chosen pair of elements of G commute. The object of this paper is to compute a precise value of commutativity degree of some finite metacyclic p -groups of class at least 3. In particular, we describe the commutativity degree of these groups in split and non-split case.

[Commutativity degree | Nilpotency class | Conjugacy class | Metacyclic group]

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1. INTRODUCTION

The commutativity degree of a finite group G is the probability that a randomly selected pairs of elements of the group commute. The concept of commutativity degree or probability of commuting pairs of a group was established by Erdos and Turan [1], Gustafson [2].

Let G be a group of finite order n . The probability $P(G)$ that two elements selected at random from G are commutative is $\frac{|\Omega|}{n^2}$ where

$$\Omega = \{(a, b) \in G \times G | ab = ba\}.$$

In order to count the elements of Ω , we have for each $a \in G$ the number of elements of Ω of the form (a, b) is $|C_G(a)|$, where $C_G(a)$ is the centralizer of a in G . Hence we have $|\Omega| = \sum |C_G(a)|$, where the sum extends over all $a \in G$. We recall that if a and b are conjugate elements of G , then $C_G(a)$ and $C_G(b)$ are conjugate subgroups. Moreover, the number of elements in the conjugacy classes of a is $[G : C_G(a)]$. Hence, if a_1, \dots, a_k are representatives of the conjugacy classes in G , then

$$|\Omega| = \sum_{i=1}^k [G : C_G(a_i)] |C_G(a_i)| = k \cdot n.$$

$$\text{Thus } P(G) = \frac{k(G)}{|G|} \quad (1)$$

where $k(G)$ is the number of conjugacy classes of G and $|G|$ is the order of G . This formula has been proved by Gustafson [2] and the method of the proof was used by Erdos and Turan [1]. Also, it has shown in [2] that if G is

any non-abelian group, then the upper bound for $\frac{k(G)}{|G|}$ is $\frac{5}{8}$ thus $P(G) \leq \frac{5}{8}$, which of course hold for all metacyclic p -groups.

Equation (1) shows that finding the commutativity degree of a group is equivalent to finding the number of conjugacy classes of the group.

There are several papers on the conjugacy classes and commutativity degree of finite p -groups see, for example [3,4,5,6,7]. Many authors achieved to significant results on the lower and upper bound for $k(G)$. For instance, Lopez in [5, Theorem 1] shows that if A is a maximal abelian subgroup of finite nilpotent group G and $|A| = p^\alpha$ then there is an integer $k \geq 0$ such that

$$k(G) = p^{2\alpha - \beta m} + \frac{p^\beta (p + 1)(p^{\alpha - \beta m} - 1)}{p^{\alpha - \beta m}} + \frac{k(p^2 - 1)(p - 1)}{p^{\alpha - \beta m}},$$

where $|G| = p^m$ and $|Z(G)| = p^\beta$.

Indeed for $k \neq 0$, this formula also shows an upper bound for G and does not determine the exact number of $k(G)$. Also, several results have been verified about conjugacy classes of subgroups of metacyclic p -groups see [8,9,10]. For example, in [10, Theorem 1.3] it was shown that if G is any finite split metacyclic p -group for an odd prime p , that is, $G = H \rtimes K$ for subgroups H and K , and if $|H| = p^\alpha$ and $|K| = p^{\alpha + \beta}$, then there exist exactly

$$\frac{(\beta - \alpha + 1)(p^{\alpha + 1} - 1)}{p - 1} + 4 \sum_{i=0}^{\alpha - 1} p^i (\alpha + i),$$

conjugacy classes of subgroups of G .

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Recently, in [11] a general formula for the exact number of conjugacy classes and commutativity degree of two generator p -groups has been computed. It has been shown that if G is a metacyclic p -group with the following presentations:

If G is nilpotent of class two, then

$$(1) G \cong \langle a, b | a^{p^\alpha} = b^{p^\beta} = 1, [a, b] = a^{p^{\alpha-\gamma}} \rangle, \\ \text{where } \alpha, \beta, \gamma \in N, \alpha \geq 2\gamma, \beta \geq \gamma \geq 1;$$

If p is odd and the class of G is greater than 2, then

$$(2) G \cong \langle a, b | a^{p^\alpha} = b^{p^\beta} = 1, [b, a] = a^{p^{\alpha-\gamma}} \rangle, \\ \text{where } \alpha, \beta, \gamma \in N, \gamma - 1 < \alpha < 2\gamma, \beta \geq \gamma;$$

$$(3) G \cong \langle a, b | a^{p^\alpha} = 1, b^{p^\beta} = a^{p^{\alpha-\varepsilon}}, [b, a] = a^{p^{\alpha-\gamma}} \rangle, \\ \text{where } \alpha, \beta, \gamma \in N, \gamma - 1 < \alpha < 2\gamma, \beta \geq \gamma, \alpha < \beta + \varepsilon;$$

then, the commutativity degree of group G is equal to

$$p^\gamma + p^{-(\gamma+1)} - p^{-(2\gamma+1)}.$$

A group G is called metacyclic if it contains a normal cyclic subgroup N such that G/N is also cyclic. The metacyclic p -groups of class 2 have been classified in context with the classification of 2-generator p -groups of class 2 for $p > 2$ in [12] and $p = 2$ in [13]. Moreover, Beuerle [14] classified the non-abelian metacyclic p -groups of class at least 3 where p is any prime.

In this paper we focus on some metacyclic 2-groups of class at least 3. Our basic goal is to compute the exact value of commutativity degree of the generalized quaternion groups, dihedral groups, semi-dihedral groups and quasi dihedral groups.

The following presentation is the generalized quaternion group $Q_{2^{\alpha+1}}$.

Theorem 1.1 [14] Let G be a metacyclic 2-group. Then

$$G \cong \langle a, b | a^{2^\alpha} = 1, b^2 = a^{2^{\alpha-1}}, [b, a] = a^{-2} \rangle, \\ \text{where } \alpha \geq 3.$$

Theorem 1.2 is a presentation of dihedral group $D_{2^{\alpha+1}}$.

Theorem 1.2 [14] If G is a metacyclic group, then

$$G \cong \langle a, b | a^{2^\alpha} = b^2 = 1, [b, a] = a^{-2} \rangle, \\ \text{where } \alpha \geq 3;$$

The following group is the semi-dihedral group $SD_{2^{\alpha+1}}$.

Theorem 1.3 [14] If G is a metacyclic 2-group, then

$$G \cong \langle a, b | a^{2^\alpha} = b^2 = 1, [b, a] = a^{2^{\alpha-1}-2} \rangle, \\ \text{where } \alpha \geq 3.$$

The group $QD_{2^{\alpha+1}}$, quasi-dihedral group has the following presentation.

Theorem 1.4 [14] If G is a metacyclic group, then

$$G \cong \langle a, b | a^{2^\alpha} = b^2 = 1, [b, a] = a^{2^{\alpha-1}} \rangle, \\ \text{where } \alpha \geq 2.$$

2. SOME BASIC RESULTS

In this section we state some results which are need to prove our theorems. First we should introduce some notations. We denote

$$G(p, \alpha, \beta, \varepsilon, \gamma) = \langle a, b | a = 1, b^{p^\beta} = a^{p^{\alpha-\varepsilon}}, a^b = a^r \rangle,$$

where $r = p^{\alpha-\gamma} + 1$. We use the notation $[b, a] = bab^{-1}a^{-1} = a^b a^{-1}$ for the commutator of b and a .

Lemma 2.1 Let α, β, r and ε be integers with α, β non-negative and let

$$G \cong \langle a, b | a^{p^\alpha} = 1, b^{p^\beta} = a^{p^{\alpha-\varepsilon}}, a^b = a^r \rangle, \\ \text{be a metacyclic } p\text{-group, where } r = p^{\alpha-\gamma} \pm 1. \text{ If } a, b \in G \\ \text{with } a = x^i y^j \text{ and } b = x^s y^t, \text{ then the following hold in } G:$$

- (i) $ab = x^{i+sr^j} y^{j+t},$
- (ii) $a^b = x^{s(1-r^j)+ir^t} y^j,$
- (iii) $[a, b] = x^{i(1-r^t)+s(r^j-1)}.$

Proof. Since $x^y = x^r$, we get $x^{y^j} = x^{r^j}$ and so $x^{iy^j} = x^{ir^j}$. Hence $y^j x^i = x^{ir^j} y^j$ and the result follows. ■

Lemma 2.2 [11] Let G be a metacyclic p -group of type

$$G(p, \alpha, \beta, \varepsilon, \gamma) = \langle a, b | a^{p^\alpha} = 1, b^{p^\beta} = a^{p^{\alpha-\varepsilon}}, a^b = a^r \rangle, \\ \text{where } r = p^{\alpha-\gamma} + 1. \text{ Then}$$

- (i) $|G| = p^{\alpha+\beta};$
- (ii) $Z(G) = \langle a^{p^\gamma}, b^{p^\gamma} \rangle;$
- (iii) $|Z(G)| = p^{\alpha+\beta-2\gamma}.$

Lemma 2.3 [15, Proposition 4.10] Let G be a metacyclic 2-group of type

$$G(2, \alpha, \beta, \varepsilon, \gamma) = \langle a, b | a^{2^\alpha} = 1, b^{2^\beta} = a^{2^{\alpha-\varepsilon}}, a^b = a^t \rangle, \\ \text{where } r = 2^{\alpha-\gamma} - 1. \text{ Then}$$

- (i) $|G| = 2^{\alpha+\beta};$
- (ii) $Z(G) = \langle a^{2^{\alpha-1}}, b^{2^{\max\{\alpha, \gamma\}}} \rangle;$
- (iii) $|Z(G)| = 2^{\beta - \max\{1, \gamma\} + 1}.$

If G is a metacyclic p -group of order $p^{\alpha+\beta}$ with relation $ba = a^r b$, then each element in the group can be written in the unique form $a^s b^t$, where all possible value for s and t is $0 \leq s < p^\alpha, 0 \leq t < p^\beta$.

Now we are ready to find the value of commutativity degree of split and non-split metacyclic 2-group of classes at least 3.

Lemma 2.4 [14] Consider a group of type $G(p, \alpha, \beta, \varepsilon, \gamma)$, then

- (i) The class of G is greater than 2 if and only if $\alpha < 2\gamma$,
- (ii) If $\beta + \varepsilon \leq \alpha$, then G is isomorphic to a split metacyclic p -group and in particular, $G \cong G(p, \alpha, \beta, 0, \gamma)$.

The following two corollaries show that when a classification of metacyclic 2-group is a split group, and when it has class 2 or greater than 2. For a proof, we refer to above lemma and [15].

Corollary 2.5 [14] *Let G be a group of type $G(2; \alpha, \beta, 1, \gamma)$, where $t = 2^{\alpha-\gamma} - 1$. If $\beta = \gamma$, then G is isomorphic to a split metacyclic 2-group and in particular, $G \cong G(2; \alpha, \beta, 0, \beta)$. Moreover, the class of G is greater than 2 if and only if $\alpha > 2$.*

Corollary 2.6 [14] *Let G be a group of type $G(p; \alpha, \beta, \varepsilon, \gamma)$ where $r = p^{\alpha-\gamma} + 1$. If $\beta + \varepsilon \geq \alpha$, then G is isomorphic to a split metacyclic p -group and in particular, $G \cong G(p; \alpha, \beta, 0, \gamma)$. Moreover, the class of G is greater than 2 if and only if $\alpha < 2\gamma$.*

In the following four theorems we give a formula for $P(G)$ in terms of α .

3. MAIN THEOREMS

Theorem 3.1 *Let G be a metacyclic 2-group of nilpotency class at least 3. If*

$$G \cong \langle a, b | a^{2^\alpha} = b^2 = 1, [b, a] = a^{2^{\alpha-1}-2} \rangle,$$

where $\alpha \geq 3$, then

$$P(G) = \frac{2^{\alpha-1} + 3}{2^{\alpha-1}}.$$

Proof. This group is generalized quaternion group $Q_{2^{\alpha+1}}$. By Lemma 2.3 the order of G is $2^{\alpha+1}$ and $Z(G) = \langle a^{2^{\alpha-1}}, b^2 \rangle$ and $|Z(G)| = 2$. We can write an arbitrary element of G in the unique form a^i and $a^i b$ where $1 \leq i \leq 2^\alpha - 1$. From relation $[b, a] = a^{-2}$, we have $ba = a^{-1}b$ so any element of the form $b^j a^i$ can be written as the following case:

$$b^j a^i = \begin{cases} a^i b^j, & j \text{ even} \\ a^{-i} b^j, & j \text{ odd.} \end{cases}$$

First we consider an element of the form $a^i \in G$ to conjugate by element a^t

$$(a^i)^{a^t} = a^t a^i a^{-t} = a^i$$

and

$$(a^i)^{a^t b} = a^t b a^i b a^{-t} = a^t a^{-(t+i)} b = a^{-i}.$$

Thus

$$[a^i] = \{a^i, a^{-i}\}.$$

In this case we have $2^\alpha - 2$ non-central elements which are partitioned into 2 element conjugacy classes of the form $[a^i]$. Consequently we have $(2^\alpha - 2)/2 = 2^{\alpha-1} - 1$ such classes. Next conjugate an element of the second form $a^i b$ by a^t and $a^t b$ then

$$(a^i b)^{a^t} = a^t a^i b a^t = a^{t+i} a^t b = a^{2t+i} b$$

and

$$(a^i b)^{a^t b} = a^t b a^{i-t} = a^t a^{t-i} b = a^{2t-i} b.$$

Thus

$$[a^i b] = \{a^{2t+i} b, a^{2t-i} b | 0 \leq t \leq 2^\alpha - 1\}.$$

Consequently,

$$[ab] = \{ab, a^3 b, \dots, a^{2^\alpha-1} b\} = \{a^i b | i \text{ is odd}\}$$

and

$$[a^2 b] = \{b, a^2 b, \dots, a^{2^\alpha-2} b\} = \{a^i b | i \text{ is even}\},$$

are 2 conjugacy classes including of all elements of the form $a^i b$ with $1 \leq i < 2^\alpha$. Thus all of non-central elements of G are consisted in one of the classes mentioned above. In addition,

$$Z(G) = \langle a^{2^{\alpha-1}}, b^2 \rangle = \langle a^{2^{\alpha-1}} \rangle$$

and $|Z(G)| = 2$. Therefore, the number of conjugacy classes of G is equal to $2^{\alpha-1} + 3$ and then

$$P(G) = \frac{2^{\alpha-1} + 3}{2^{\alpha-1}}. \quad \blacksquare$$

Theorem 3.2 *Let G be a metacyclic group presented by $G \cong \langle a, b | a^{2^\alpha} = b^2 = 1, [b, a] = a^{-2} \rangle$, where $\alpha \geq 3$. Then*

$$P(G) = \frac{2^{\alpha-1} + 3}{2^{\alpha+1}}.$$

Proof. This group is the dihedral group $D_{2^{\alpha+1}}$ of order $2^{\alpha+1}$. The group G is split extension and by Lemma 2.3, $Z(G) = \langle a^{2^{\alpha-1}} \rangle$ and $|Z(G)| = 2$. From the relation $[b, a] = a^{-2}$, we have $ba = a^{-1}b$. We can write

$$G = \{a^i | 0 \leq i \leq 2^\alpha - 1\} \cup \{a^i b | 0 \leq i \leq 2^\alpha - 1\}.$$

We now conjugate elements of G by a^i and $a^i b$ to find the conjugacy classes. Thus any element of the form $b^j a^i$ can be written by $b^j a^i = a^{i(-1)^j} b^j$. Suppose $0 \leq t \leq 2^\alpha - 1$ then

$$(a^t)^{a^i} = a^i a^t a^{-i} = a^t$$

and

$$(a^t)^{a^i b} = a^i b a^t b a^{-i} = a^i b a^{t+i} b = a^{-t} b^2 = a^{-t}.$$

Thus

$$[a^t] = \{a^t, a^{-t}\}.$$

We suppose that $t \neq 2^{\alpha-1}, 0$. Then $2^{\alpha} - 2$ non-central elements are partitioned into two element conjugacy classes of the form $[a^t] = \langle a \rangle$. Hence we have $(2^{\alpha} - 2)/2 = 2^{\alpha-1} - 1$ such classes. Next we conjugate b by a^{-j} and $a^j b$, then we have

$$(b)^{a^{-j}} = a^{-j} b a^j = a^{-2j} b$$

and

$$(b)^{a^j b} = a^j b a^{-j} = a^{2j} b.$$

Hence

$$[b] = [a^{2j} b] = \{a^{2j} b \mid 0 \leq j \leq \frac{2^{\alpha}-1}{2}\}.$$

Here half of the elements of the form $a^i b$ are included by $[b]$. We should find all further conjugacy classes including elements of the form $a^i b$ establishing with $[ab]$. Suppose $0 \leq j \leq 2^{\alpha} - 1$ then

$$(ab)^{a^{-j}} = a^{-j} a b a^j = a^{1-2j} b,$$

and

$$(ab)^{a^{-j} b} = a^{-j} b a b^2 a^{-j} = a^{2j-1} b.$$

Since

$$\langle a^{2j+1} \rangle = \{a^i : 0 \leq i < 2^{\alpha} \text{ and } i \text{ is odd}\},$$

then

$$[ab] = [a^{2j+1} b] = \{a^i b : 0 \leq i < 2^{\alpha} \text{ and } i \text{ is odd}\}.$$

Hence $[ab]$ includes the other half of the elements of the form $a^i b$. Therefore all classes containing elements of the form $a^i b$ with $0 \leq i \leq 2^{\alpha} - 1$ have been found. Hence we have 2 conjugacy classes of the form $[ab]$ and $[a]$. All non-central elements of G are included in one of the classes expressed above. As mentioned before we have

$$Z(G) = \langle a^{2^{\alpha-1}}, b^2 \rangle = \langle a^{2^{\alpha-1}} \rangle.$$

So $Z(G)$ contains $|Z(G)| = 2$ conjugacy classes. Therefore we have

$$k(G) = 2 + 2^{\alpha-1} + 1 = 2^{\alpha-1} + 3.$$

Hence

$$P(G) = \frac{2^{\alpha-1} + 3}{2^{\alpha+1}}. \quad \blacksquare$$

Theorem 3.3 Let G be a metacyclic 2-group. If

$$G \cong \langle a, b \mid a^{2^{\alpha}} = b^2 = 1, [b, a] = a^{2^{\alpha-1}-2} \rangle,$$

where $\alpha \geq 3$, then

$$P(G) = \frac{2^{\alpha-1} + 3}{2^{\alpha+1}}.$$

Proof. This group is semi-dihedral group $SD_{2^{\alpha+1}}$ with order $2^{\alpha+1}$. Also,

$$Z(G) = \langle a^{2^{\alpha-1}}, b^2 \rangle = \langle a^{2^{\alpha-1}} \rangle$$

and $|Z(G)| = 2$. Each element of the group G can be written in the unique form a^s or $a^s b$ with $0 \leq s < 2^{\alpha}$. We rewrite $[b, a] = a^{2^{\alpha-1}-2}$ to $ba = a^r b$ where $r = 2^{\alpha-1} - 1$. We now conjugate elements of G to find conjugacy classes. Suppose $a^i \in G$. Then conjugate a^t by a^i and $a^i b$ thus we have

$$(a^t)^{a^i} = a^i a^t a^{-i} = a^t.$$

We now conjugate a^t by $a^i b$. By using Lemma 2.1 and for $1 \leq t < 2^{\alpha}$ we have

$$\begin{aligned} (a^t)^{a^i b} &= a^i b a^t a^{-ir} b \\ &= a^i b a^{t-ir} b \\ &= a^{i+(t-ir)r} \\ &= a^{i+(t-i(2^{\alpha-1}-1))(2^{\alpha-1}-1)} \\ &= a^{t2^{\alpha-1}-t}. \end{aligned}$$

If t is even, then a^t is a central element and $[a^t] = \{a^t, a^{-t}\} = \{a^t\}$. If t is odd then

$$[a^t] = \{a^t, a^{2^{\alpha-1}-t}\}$$

contains $\frac{2^{\alpha}-2}{2} = 2^{\alpha-1} - 1$ conjugacy classes of order two. Next we conjugate $a^t b$ by a^i and $a^i b$ respectively. Thus for $1 \leq t < 2^{\alpha}$

$$\begin{aligned} (a^t b)^{a^i} &= a^i a^t b a^{-i} \\ &= a^{i(1-r)+t} b \\ &= a^{2i-i2^{\alpha-1}+t} b \end{aligned}$$

and

$$\begin{aligned} (a^t b)^{a^i b} &= a^i b a^t b b^{-1} a^{-i} \\ &= a^i a^{(t-i)r} b \\ &= a^{i+(t-i)(2^{\alpha-1}-1)} b \\ &= a^{(t-i)2^{\alpha-1}+2i-t} b. \end{aligned}$$

Therefore,

$$[a^t b] = \{ a^{2i-2\alpha-1+t} b, a^{(t-i)2\alpha-1+2i-t} b \mid 0 \leq i \leq 2^\alpha - 1 \}$$

$$= \{ a^t b, a^{2\alpha-1-t} b, \dots, a^{2\alpha-2\alpha-2+t} b, a^{(t-2\alpha-1)2\alpha-1+2\alpha-t} b \}.$$

For $t = 1$, we have

$$[ab] = \{ ab, a^{2\alpha-1} b, \dots, a^{2\alpha-2\alpha-2+1} b, a^{(1-2\alpha-1)2\alpha-1+2\alpha-1} b \}$$

$$= \{ a^k b \mid k \text{ is odd} \}.$$

For $t = 2$, we have

$$[a^2 b] = \{ a^2 b, a^{2\alpha-1-2} b, \dots, a^{2\alpha-2\alpha-2+1} b, a^{(2-2\alpha-1)2\alpha-1+2\alpha-2} b \}$$

$$= \{ a^k b \mid k \text{ is even} \}.$$

Thus there are two conjugacy classes with $2^{\alpha-1}$ element.

On the other hand, $Z(G) = \langle a^{2^{\alpha-1}}, b^2 \rangle = \langle a^{2^{\alpha-1}} \rangle$ contains $|Z(G)| = 2$ conjugacy classes. Therefore we have

$$P(G) = \frac{2^{\alpha-1}+3}{2^{\alpha+1}}. \quad \blacksquare$$

Theorem 3.4 Let G is a metacyclic 2-group and

$$G \cong \langle a, b \mid a^{2^\alpha} = b^2 = 1, [b, a] = a^{2^{\alpha-1}} \rangle,$$

where $\alpha \geq 2$. Then $P(G) = \frac{5}{8}$.

Proof. This group is the quasi-dihedral group $QD_{2^{\alpha+1}}$ of order $2^{\alpha+1}$. Using Lemma 2.2, $Z(G) = \langle a^2 \rangle$ and $|Z(G)| = 2^{\alpha-1}$. By Corollary 2, this group is a split group of class greater than 2. We obtain $k(G)$ by computing the number of x^G for $x \in G$. Note that an arbitrary element of G can be written uniquely in the form

$$G = \{ a^i b^j \mid 0 \leq i < 2^\alpha, 0 \leq j < 2 \}.$$

Also, $Z(G) = \langle a^2, b^2 \rangle = \langle a^2 \rangle$. Moreover, from Lemma 2.1 we have

$$(a^i b^j)^{a^s b^t} = a^s b^t a^i b^j b^{-t} a^{-s} = a^{s(1-r^j)+ir^t} b^j,$$

where $r = 2^{\alpha-1} + 1$. Since $b^2 = 1$, it is convenient to work with two forms a^k and $a^k b$. Hence we can apply again Lemma 2.1 to find the $|x^G|$ for some $x \in G$. Thus

$$(a^i)^{a^s b} = a^s b a^i b a^{-s} = a^{s+ir-sr^2} = a^{i(2^{\alpha-1}+1)},$$

because $|a| = 2^\alpha$. Similarly $(a^i)^{a^s} = a^i$. Hence

$$[a^i] = \{ a^i, a^{i(2^{\alpha-1}+1)} \}.$$

If i is even then $a^i \in Z(G)$ and $[a^i]$ is the singleton $\{a^i\}$. If i is odd then

$$[a^i] = \{ a^i, a^{2^{\alpha-1}+i} \}.$$

In this case we have $\frac{2^\alpha/2}{2} = 2^{\alpha-2}$ conjugacy classes of order 2. Likewise, we have

$$(a^i b)^{a^s} = a^{s(1-r)} a^i b = a^{i-s2^{\alpha-1}} b$$

and

$$(a^i)^{a^s b} = a^{s(1-r^j)+ir^t} b^j = a^{(s+i)(2^{\alpha-1})+i} b.$$

Thus

$$[a^i b] = \{ a^i b, a^{(2^{\alpha-1}+i)} b \}.$$

In this case we have $2^{\alpha-1}$ conjugacy classes with 2 elements. All non-central elements of G are included in one of the classes mentioned above. Also, $Z(G) = \langle a^2 \rangle$ contains $|Z(G)| = 2^{\alpha-1}$ conjugacy classes. Hence we have

$$k(G) = 2^{\alpha-2} + 2^{\alpha-1} + 2^{\alpha-1} = 2^\alpha + 2^{\alpha-2}$$

$$P(G) = \frac{2^{\alpha-1}+2^{\alpha-2}}{2^{\alpha+1}} = \frac{5}{8}. \quad \blacksquare$$

4. CONCLUSION

The commutativity degree of dihedral groups, semi-dihedral groups and quasi-dihedral groups are the same.

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